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# A Class of Planar Discrete Velocity Models for Gas Mixtures

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We present a method for the construction of a simple class of physically acceptable planar discrete velocity models (DVMs) for binary gas mixtures. We want five conservation laws (no more, no less) with binary collisions. We first consider a collision with a particle at rest and different possibilities for the three other particles. We associate other particles and find semisymmetric $qv_i$ models with $q = 7$, 9, 11, 13 and 15, symmetric with respect to the two coordinate axes, but not to an exchange between the two axes. In order to avoid "spurious" mass conservation relations for the species without particle at rest, we find, for the two coordinate axes, that *the tips of the momenta of the particles must be on two intervals parallel to one axis with opposite values on the other*. There remain some physically acceptable $q = 9$ (the smallest) and 11, 13, 15 models (adding multiple collisions for some others). Second, we construct the associated symmetric models $qv_i \cup \hat{q}v_i$, which are superpositions of the $qv_i$ model and another $\hat{q}v_i$, rotated by $\pi/2$. The possible previous defect of the spurious mass invariant for $qv_i$ is transmitted to the symmetric one. We explain another defect coming from $qv_i$ and $\hat{q}v_i$ having only one common particle, *then "spurious" invariants exist for the momentum conservations along the two axes*. We get four physically acceptable symmetric $17v_i$ (and three intermediate semisymmetric $13v_i$ models) and one $25v_i$ model superposition of two $11v_i$ and two $15v_i$ models (other acceptable symmetric $11v_i$, $13v_i$, and $25v_i$ models exist with multiple collisions).

**KEY WORDS:** Discrete velocity models; mixtures, collision invariants.

## 1. INTRODUCTION

Recently Bobylev and Cercignani[1] introduced the idea that Discrete Velocity Models (DVMs) for gas mixtures in a wide context are objects of mathematical and physical interest. They proposed a general method to

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construct planar binary mixture models with masses equal to 1, $M$ for the light and heavy species. As illustration, with binary collisions, they described two specific simple models, symmetric with respect to an exchange between the two axes: $13v_i$, $M=2, 5$ and $25v_i$, $M=2, 5$.

One of us (H.C.), for the two first symmetric $13v_i$, $M=2, 5$ models, found two spurious invariants which will be discussed in the present paper (a way to eliminate these spurious invariants is to introduce multiple collisions). He found also, with only binary collisions, two semisymmetric $11v_i$, $M>1$, $13v_i$, $M=5$ and one symmetric $17v_i$, $M=2$ models. *At that time they were the first known models without spurious invariants.*

In two meetings,[2] the authors of ref. 1 mentioned both this defect for their $13v_i$ symmetric models and that other people have also found spurious invariants for their two symmetric $25v_i$, $M=2, 5$ models.

Cercignani and Cornille[3] have presented the three above models with $11v_i$, $13v_i$, $17v_i$ velocities and studied the properties (shock waves) for the two first ones. These planar models have 5 physical conservation laws as expected for a binary mixture in the plane case. For the construction they first considered a collision with a heavy rest particle and one of the three other particles with momenta $(x, y)$ being along one coordinate axis. They associated other collisions with momenta $(\pm x, \pm y)$ and they obtained a $11v_i$ model with $M$ arbitrary, not symmetric with respect to the exchange of the $x$ and $y$ coordinates. They deduced another semisymmetric $13v_i$, $M=5$ and the symmetric $17v_i$, $M=2$ model with $(y, x)$ momenta associated to $(x, y)$. Finally to these models they added another class of $11v_i$, $13v_i$, $17v_i$ models with a particle at rest of the light species. To their semisymmetric $11v_i$ models they associated the rotated by $\pi/2$, $\widehat{11}v_i$ models satisfying $11v_i \cap \widehat{11}v_i = 5v_i$ and, from $11v_i \cup \widehat{11}v_i = [11+11-5]\, v_i$, they got two symmetric $17v_i$ models.

Recently Bobylev and Cercignani[4] discussed a method of constructing models without spurious invariants. They mentioned that the minimal semisymmetric models were our previous $11v_i$ models[3] and that their study confirms that their symmetric $25v_i$, $M=2, 5$ models[1] have spurious invariants.

In the present paper, we extend the method of ref. 3 for the construction of models without spurious invariants and, for instance, find two results different from those of ref. 4. We find that the smallest semisymmetric model is $9v_i$ (not $11v_i$) and for the $25v_i$ models, that the $M=2$ model, with binary collisions, has no spurious invariants (for $M=5$ we add multiple collisions).

We have two motivations for the present work. On the one hand we generalize the method of ref. 3, following the same steps, with different possibilities depending on the momenta of the particles in the original collision with a particle at rest. On the other hand a difficulty for the construction of such models is the possibility to have "spurious invariants" that we call "virus," for instance *two mass conservations for one species or two momentum conservations along one axis*. We want to understand how such virus can appear and if a possible medicine to eliminate it exists. However for the species including the particle at rest, it is easy to prove that such a virus for the mass conservation cannot exist.

In the first step we construct the collision with exchange of energy including one rest particle (either heavy or light). For the three momenta associated to the three other particles we have different possibilities: either two of them are along the coordinate axes, or only one or finally zero. Then, to the $(x, y)$ momentum components of the previous three particles we associate other particles with momenta $(\eta_1 x, \eta_2 y)$, $\eta_i^2 = 1$. In this way we obtain three classes of $qv_i$ semisymmetric models with respectively $q = 9$, 11 and 13 particles, presented in Figs. 1, 2a, b, and 3. The study is done in Section 2.

These semisymmetric models are symmetric with respect to the $x$, $y$-axes but not to an exchange of the $y$ and $x$ axes. For the physically acceptable models (without virus with only binary collisions) we find that the tips of the momenta are both along two intervals parallel to the $x$ axis with opposite values for the $y$-component, $y = \pm v$ (with $v$ integer only for the $11v_i$ model with $M = 2$ or 5 and along two intervals parallel to the $y$ axis with opposite values of the $x$-component, $x = \pm 1$. If this geometrical structure is not satisfied we find a virus coming from a possible "spurious relation" for the mass conservation of the species without a particle at rest. We give an example, cf. Fig. 2c, not satisfying completely the above geometrical structure, with an arbitrary parameter. There are tips of the momenta parallel on two $x$-axis intervals but not on the $y$-axis and consequently we find, for the species without particle at rest, two mass conservations (a medicine for particular values of the arbitrary parameter being possible with multiple collisions). They reduce to only one for the particular value of the arbitrary parameter recovering the geometrical property for the $y$-axis and the model of Fig. 2a. However even in this case we notice a particular case without spurious conservation laws (only 5 conservation equations) but with an ambiguous conservation relation for the mass of the species without the particle at rest. Later we present the three acceptable classes giving the $9v_i$, $11v_i$, $13v_i$ models.

In the second step, for all previous $qv_i$ models we can, with a rotation of $\pi/2$, associate other ones $\hat{q}v_i$, the intervals parallel to the axes with $y = \pm v$ or $x = \pm 1$ becoming $y = \pm 1$ or $x = \pm v$. In Section 3 we obtain symmetric $qv_i \cup \hat{q}v_i$ models with $17v_i$, $21v_i$, $25v_i$ particles. However a great difference occurs depending on whether $v$ is not an integer (starting with $9v_i$ models, part of the $11v_i$ models and $13v_i$ models) or is integer, $v=1, 2$ for the $M=5$, 2, $11v_i$ models. For the models with $v \neq 1$, $\neq 2$, only the particle at rest is common to the two starting models. Then the momentum relation along one axis is the sum of the momentum relations for the two building-up models and a virus leading to a "spurious momentum invariant" exists. This virus can be detected in the original $qv_i$ model, depending on whether it has momenta along the bisectors of the two axes or not. On the contrary for the $11v_i$ models with $v=1, 2$, leading to $17v_i$ models, five particles are common and only one momentum conservation exists for each axis. In this way (Figs. 4) we find four acceptable models: two with $M=2$ (light or heavy (ref. 3) particle at rest) and two other ones with $M=5, 3$ (heavy or light particle at rest).

In Section 4, starting with a collision term with exchange of energy along the $x$-axis, we obtain (Fig. 5) both $7v_i$ and symmetric planar $7v_i \cup \widehat{7}v_i = 13v_i$ models (cf. ref. 1 for $M=2, 5$). With binary collisions both models have spurious mass conservations (disappearing with multiple collisions).

An extension of the present method is to start with more than one collision term with exchange of energy, for instance the $25v_i$ model of ref. 1. In Section 5 and Figs. 6–7, starting with two collision terms, (giving three collisions), we get $qv_i = 15v_i$ and $25v_i = 15v_i \cup \widehat{15}v_i$ models for $M \geqslant 2$. We have not ($15v_i \cap \widehat{15}v_i = 5v_i$) the virus leading to spurious momentum relation. What about the virus associated to the tips of the momenta not parallel to the $y$-axis?. For $M \neq 2$ (contrary to $M=2$) we have particular tips of momenta not on two intervals parallel to the $y$-axis, and we recover the same virus as for some previous examples of the $11v_i$, $13v_i$ models. We find for both the $15v_i$ and the $25v_i$ models, spurious mass invariants for the species without particle at rest (a medicine being multiple collisions).

We write our proofs in different lemmas and write **As Illustration** when these lemmas can be verified with explicit collision terms. For these gas mixtures we call $f_i$ $(\vec{p}_i)$, $l_i$ the densities (momenta) and left hand side of the evolution equations for the light particles with mass equal to 1 and $F_i$ $(\vec{P}_i)$, $L_i$ for the heavy particles with mass equal to $M$. *In order to check physical models* (*or not*) we present a pedestrian, but efficient, method. For instance for the mass of the species without $F_0$ we consider only the collisions including $f_i$. Starting with one such collision we write the most general linear combination of $l_i$ not containing it, then do the same with a second, third ... collision and get $\sum a_i l_i = 0$. In this sum either all the constants $a_i$ are equal and all $l_i$ are present or not. In the second case we have spurious mass invariants. We look at the missing collisions (with a geometrical explanation) and eliminate the useless collisions. In this way, in Section 6, for the $25v_i$, $M=2$ and $M \neq 2$ models (46 and 44 collisions), only 11 and 10 collisions are sufficient to detect the virus of the spurious invariant.

For any set $X_i$, $x_i$ we define $X_{i,\,j,\dots,\,p} = \sum_{s=i}^{s=p} X_s$, $x_{i,\,j,\dots,\,p} = \sum_{s=i}^{s=p} x_i$.

## 2. SEMISYMMETRIC $qv_i = 9v_i$, $11v_i$, $13v_i$ PLANAR MODELS

We present in Figs. 1, 2a, b, and 3 the $9v_i$, $11v_i$, $13v_i$ models, deduced from the starting collisions $f_0 F_i - f_j F_k$ or $F_0 f_i - F_j f_k$ with an exchange of energy between the two species and two, one or zero particles along the coordinate axes.

**Lemma 1.** No spurious mass invariant (for the $qv_i$ and symmetric $qv_i \cup \hat{q}v_i$ models) for the species associated to the particle at rest $f_0$ (or $F_0$).

The important point is that all other particles of this species are linked to the particle at rest by a collision with exchange of energy between the two species: $\Gamma_i^{exch} = F_k f_0 - F_m f_i$ (or $\Gamma_i^{exch} = F_0 f_k - F_i f_m$). Let us start with the light-species densities $f_0$, $f_i$ and $l_i$, $l_0$. For the density of the particle at rest $f_0$ we have $l_0 = -\sum_i \Gamma_i^{exch}$. In order to have a linear relation between the $l_i$ we must successively add $l_1$, $l_2$,... with all $l_i$ and with only these collision terms with energy exchange we have $l_0 + \sum_{i \neq 0} l_i = 0$. Let us recall that

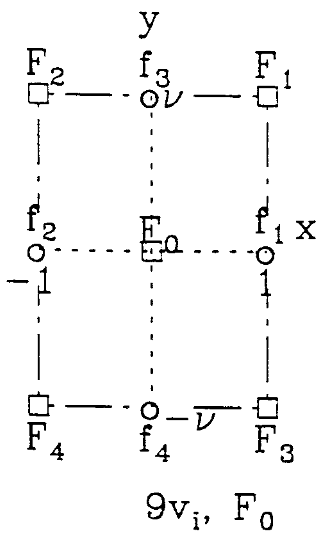



Fig. 1. $9v_i$ model with $F_0$ heavy rest-particle and $M \geqslant 2$.

the light or heavy mass conservation is satisfied for any collision. Consequently the other collisions including $l_i$ but not $f_0$, $l_0$: collisions of the type $f_k f_l - f_i f_m$ or $F_k f_l - F_m f_i$ give zero in the $\sum_{i \neq 0} l_i$ (they give opposite contributions in the loss and gain terms). The same proof holds for $F_0$, $F_{j \neq 0}$.

We always have the total mass conservation for the sum of the two species and we deduce, from Lemma 1, that either $f_0$, $\sum L_i = 0$ or $F_0$, $\sum l_i = 0$. However it could happen that there are two subsets in this species, satisfying separately a mass conservation relation, *for instance if we detect two particles of the species, without rest particle, which cannot be connected with binary collisions*. Such a virus of spurious mass invariant does not exist for the models in Figs. 1, 2a, b, and 3. We present in Subsection 2.1, Fig. 2c, another $11v_i$ model explaining how this virus can appear and the way to eliminate it, Fig. 2a.

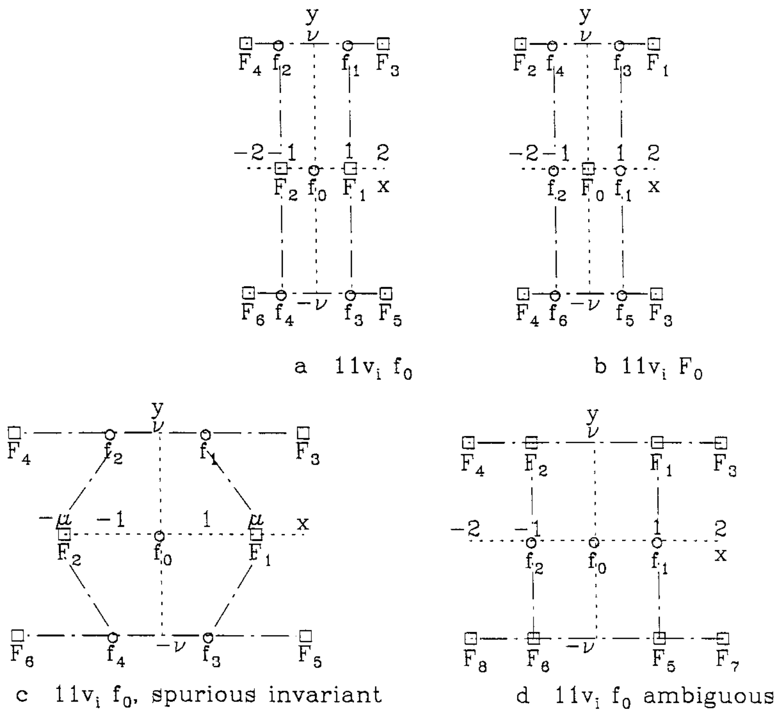



Fig. 2. (a) $11v_i$ model with $f_0$ light rest-particle and $M=2$. (b) $11v_i$ model with $F_0$ heavy rest-particle and $M \geqslant 2$. (c) $11v_i$ model with $f_0$ light rest-particle, $M=2$ and a spurious heavy mass conservation. (d) $11v_i$ model with $f_0$ light rest-particle, $M=3$ and ambiguous heavy mass conservation.

## 2.1. $11v_i$ with Possible Spurious Mass Invariants (Figs. 2)

(i) *We start with three classes of collisions* with energy exchange:

$$f_0F_3 - f_1F_1, \vec{P}_3(1+\mu, v) = \vec{p}_1(1, v) + \vec{P}_1(\mu, 0), (M-1)\, v^2 = 1 + 2\mu - M,$$

$$\mu = 1 \to v^2 = (3-M)/(M-1),\ 1 < M < 3 \to \mu = 1,\ M = 2,\ v = 1 \quad (2.1)$$

$$F_0f_3 - F_1f_2\colon \vec{p}_3(1, v) = \vec{p}_2(-\mu, 0) + \vec{P}_1(1+\mu, v),\ v^2(M-1) = (1+\mu)^2$$

$$+ M(\mu^2 - 1),\ \mu = 1 \to v^2 = 4/(M-1) \to M = 5,$$

$$v = 1 \text{ and } M = 2,\ v = 2 \quad (2.2)$$

$$f_0F_3 - F_1f_1\colon \vec{P}_3(2, v) = \vec{p}_1(1, 0) + \vec{P}_1(1, v),\ M = 3,\ v \text{ arbitrary} \quad (2.3)$$

To the momenta $(x, y)$ we associate 7 momenta $(\eta_1 x, \eta_2 y)$, $\eta_i^2 = 1$, We obtain $11v_i$ models associated to (2.1), $\mu = 1$ (Fig. 2a), $\mu \neq 1$ (Fig. 2c), to (2.2), $\mu = 1$ (Fig. 2b), and to (2.3) (Fig. 2d). We find 10 collision terms and 6 linear relations in Fig. 2c and 12 collisions with five linear relations in the other case. *However, without a detailed knowledge of these collisions, three invariants* (which give physical invariants by linear combinations), can be deduced from the geometry of these models. For the $11v_i$, $9v_i$, Figs. 1–2 all momenta are parallel or along the $x$-axis with $y$ components $v, 0, -v$. Let us call $X_v, X_0, X_{-v}$, the associated sum of the left-hand sides of the evolution equations (for instance $X_v = L_{3,4} + l_{1,2}$, $X_0 = l_0 + L_{1,2}$, $X_{-v} = L_{5,6} + l_{3,4}$ in Figs. 2a–c).

**Lemma 2.** We prove that $X_v = X_{-v} = X_0 = 0$.

First $\sum L_i = \sum l_i = 0$ giving $X_{v,-v} + X_0 = 0$ from Lemma 1 and $X_v = X_{-v}$ (momentum conservation along the $y$-axis). Second, for the (2.1) models, we can write the energy relation as $(1+v^2)\, X_{v,-v} = -\mu^2 \sum L_i/M = 0$, for the (2.2) (Fig. 2b) model as $(1 + v^2 - \mu^2) \sum_{v,-v} = -\mu^2 \sum l_i = 0$ and for the model (2.3) as $X_{v,-v} = \sum L_i = 0$. We deduce $X_{\pm v} = X_0 = 0$.

In general these invariants $X_v, X_{-v}, X_0$ have no physical meaning, however by linear combinations we have seen that we recover 4 physical invariants: momentum along the $y$-axis, mass conservations and energy. It remains only to check the momentum along $x$ and to verify that the mass conservation for the species without particle at rest has no spurious invariant.

In Fig. 2d $X_0 = 0$ is the mass conservation $\sum l_i = 0$ whereas $X_v = L_{1,2,3,4} = 0$ and $X_{-v} = L_{5,6,6,7} = 0$ are the sums over two subsets of $\sum L_i = 0$. Although there are only five linear relations in this model, we do not retain two mass relations for the same species. *We call "ambiguous" such a model.*

**Lemma 3.** For the $\mu \neq 1$ models, defined in (2.1), $f_0$ (Fig. 2c) and (2.2), $F_0$ we have not the tips of momenta parallel to the $y$-axis with opposite values ($x = \pm 1$), no collisions with sums of momenta along the $y$-axis and there exists a spurious mass relation for the species without $f_0$ (or $F_0$).

We have 6 $F_i$ (or $f_i$) and 4 collisions with exchange of energy with pairs of $F_i$ ($f_i$) leading to two triplets $L_{i,j,k} = 0$ ($l_{i,j,k} = 0$). In each triplet we have two particles with the same $x$ values and opposite $y = \pm v$ values and for the four collisions with sums of momenta along the $x$-axis they are in loss and gain terms, still giving the same $L_{i,j,k} = 0$ ($l_{i,j,k} = 0$). For the collisions with opposite momenta, for each triplet we still have vanishing contributions coming from the loss and gain terms. Finally we have two $L_{i,j,k} = 0$ ($l_{i,j,k} = 0$) and one spurious mass invariant for the species without particle at rest. Here (contrary to the $13v_i$, see later), the collisions with sums of momenta along the $x$-axis are useless while such collisions along the $y$-axis are missing.

**As Illustration** we write the 10 collisions with $i = 1, 2$ only for Fig. 2c, $\mu \neq 1$:

$$\begin{gathered}
\Gamma_i = c(f_0 F_{i+2} - f_i F_i), \qquad \Gamma_{2+i} = c(f_0 F_{4+i} - f_{2+i} F_i) \\
\Gamma_{4+i} = \bar{c}(F_{3+i} f_{3+i} - F_{4+i} f_i) \\
\Gamma_{6+i} = \tilde{c}(F_{2+i} f_{5-i} - F_{4+i} f_{3-i}), \qquad \Omega = b(F_3 F_6 - F_4 F_5) \\
\Lambda = a(f_1 f_4 - f_2 f_3)
\end{gathered} \tag{2.4}$$

We see that the two subsets ($F_1, F_3, F_5$), ($F_2, F_4, F_6$) are, in all collisions where they are present, both in the loss or the gain terms, and we verify Lemma 3: $L_{1,3,5} = 0 = L_{2,4,6}$. Similary for $\mu \neq 1$, (2.2), we verify $l_{1,3,6} = l_{2,4,5} = 0$ from 10 collision terms. We verify $\sum_0^4 l_i = 0$ in (2.4) or Lemma 1 and $X_{\pm v} = X_0 = 0$ or Lemma 2 and recover four physical conservation relations. It remains the momentum along the $x$-axis: $\mu(L_{1,3,5} - L_{2,4,6}) + [L_{3,5} - L_{4,6} + l_{1,3} - l_{2,4}]$ where both terms vanish in (2.4). In order to eliminate the virus leading to a spurious mass invariant we must have collision terms which do not annihilate each other in the sum over the first and second subsets. We have two medicines:

**Lemma 3bis.** The Lemma 3 spurious invariant disappears either with only binary collisions for $\mu = 1$, Figs. 2a, b and the tips of the momenta parallel to the $y$-axis or with multiple collisions for $\mu \neq 1$, $\mu$ integer. In both cases we have collisions with sums of momenta along the $y$-axis and one particle of both triplets in loss and gain terms.

**As Illustration**. (i) We begin with $\mu=1$ in (2.1) (Fig. 2a), $v=1, \neq 1$ for $M=2, \neq 2$ and add to (2.4) two new binary collisions mixing $F_1, F_2$:

$$\Gamma_9=\hat{c}(f_1F_2-f_2F_1), \qquad \Gamma_{10}=\hat{c}(f_3F_2-f_4F_1), \rightarrow \Gamma_{9,10}=L_{1,3,5}=-L_{2,4,6} \tag{2.4'}$$

We have $\vec{p}_1+\vec{P}_2=\vec{p}_2+\vec{P}_1=(0, v)$ and momenta parallel to the $y$-axis with $x=\pm 1$. Similarly for $\mu=1$ in (2.2) Fig. 2b we have new collisions $f_2F_1-f_1F_2$, $f_4F_1-f_3F_2$ mixing the two $(l_i)$ subsets. The other conservations hold so that *there are no spurious invariants for the* $11v_i$ *models with binary collisions and tips of the momenta parallel to both axes* (*opposite values*)

(ii) We go on with multiple collisions (sums of momenta along the $y$-axis) when $\mu\neq 1$ is an integer in (2.1) Fig. 2c: $f_i^\mu F_2-f_{i+1}^\mu F_1$, $i=1, 3$ (similarly in (2.2), $\mu$ integer), the spurious mass invariant disappears and we have 5 physical invariants. Finally we notice that in (2.1) with $1<M<1+2\mu$ we have physical models with $v$ not rational except for $\mu=1$, $v=1, 2$, $M=2, 5$.

## 2.2. Only Five Conservation Relations for the $9v_i$ Model (Fig. 1)

We start with a collision term with energy exchange: $F_0f_3-F_1f_2$:

$$\vec{P}_0+\vec{p}_3(0, v)=\vec{p}_2(-1, 0)+\vec{P}_1(1, v)=\vec{p}_1(1, 0)+\vec{P}_2(-1, v)\rightarrow M>1$$
$$v\neq 1 \text{ or } \neq 2, \qquad v=\sqrt{(M+1)/(M-1)}\rightarrow v=\sqrt{3} \qquad \text{for} \quad M=2 \tag{2.5}$$

and for Fig. 1, add the $(\pm x, \pm y)$ momenta. We recall $\sum L_i=\sum l_i=0$ from Lemma 1. For Lemma 2: $X_v=L_{1,2}+l_3=X_{-v}=L_{3,4}+l_4=X_0=L_0+l_{1,2}=0$, with $X_v=X_{-v}$ (momentum along $y$), it is sufficient to write the energy relation as $(1-v^2)(X_v+X_{-v})=\sum l_i=0$. It remains the momentum along the $x$-axis: $L_{1,3}+l_1=l_2+L_{2,4}$, verified with the 7 collision terms:

$$i=1, 2, \qquad \Gamma_i=\bar{c}(F_{2+i}f_3-F_if_4), \qquad \Gamma_{3i}=c(F_0f_3+F_if_{3-i}-2F_{3-i}f_i)$$
$$\Gamma_{4i}=c(F_0f_4+F_{5-i}f_i-2F_{2+i}f_{3-i}), \qquad \Omega=b(F_1F_4-F_2F_3)$$

**As Illustration**, for the set $(l_i)$ we define $\bar{\in}$ (only in) and notice $\Gamma_i\bar{\in} l_3, l_4$ is eliminated in $l_{3,4}=-\sum(\Gamma_{3i}+\Gamma_{4i})=-l_{1,2}$ giving only $l_{1,2,3,4}=0$.

### 2.3. $13v_i$, Fig. 3. Models Without Spurious Invariants

*First with a light particle at rest we write* 4 *collision terms* $f_0F_3 - f_iF_i$:

$$\vec{p}_0 + \vec{P}_3(1, 2v) = \vec{p}_2(-1, v) + \vec{P}_1(2, v) \to M = 2,\ v = \sqrt{5},\ (\text{Fig. 3a}) \tag{2.6a}$$

$$\vec{p}_0 + \vec{P}_3(2, v) = \vec{p}_1(1, -v) + \vec{P}_1(1, 2v) \to M = 2,\ v = \sqrt{1/5} \tag{2.6b}$$

$$\vec{p}_0 + \vec{P}_3(1, 2v) = \vec{p}_1(2, v) + \vec{P}_2(-1, v) \to M = 2,\ v = \sqrt{8},\ (\text{Fig. 3b}) \tag{2.7a}$$

$$\vec{p}_0 + \vec{P}_3(2, v) = \vec{p}_1(1, 2v) + \vec{P}_1(1, -v) \to M = 2,\ v = \sqrt{1/8} \tag{2.7b}$$

*Then with a heavy particle at rest we write* 4 *other collision terms* $F_0f_3 - F_if_i$:

$$\vec{P}_0 + \vec{p}_3(1, 2v) = \vec{p}_2(-1, v) + \vec{P}_1(2, v) \to v^2 = 4/(3M-1) \tag{2.8a}$$

$$\vec{P}_0 + \vec{p}_3(2, v) = \vec{p}_1(1, -v) + \vec{P}_1(1, 2v) \to v^2 = (3M-1)/4 \tag{2.8b}$$

$$\vec{P}_0 + \vec{p}_3(1, 2v) = \vec{p}_1(2, v) + \vec{P}_2(-1, v) \to v^2 = (3M+1)/(3M-1) \tag{2.9a}$$

$$\vec{P}_0 + \vec{p}_3(2, v) = \vec{p}_1(1, 2v) + \vec{P}_1(1, -v) \to v^2 = (3M-1)/(3M+1) \tag{2.9b}$$

Adding 9 new momenta with $(\pm x, \pm y)$ we get eight $13v_i$ models that, for brevity, we call (2.qa, b) models with $q = 6, 7, 8, 9$. Those with $f_0$ have only $M = 2$ while those with $F_0$ have $M > 1$. Notice the symmetries: (i) $v \to 1/v$ for (2.qa) $\to$ (2.qb). For the locations of $\vec{p}_i$, $\vec{P}_i$ they look like "models with a rotation by $\pi/2$," except for the changes $x = \pm 1 \to y = \pm 1/v$, $y = \pm v \to x = \mp 1$. (ii) Forgetting the values for $v$ there are also symmetries for the vectors $\vec{p}_i \rightleftharpoons \vec{P}_i$ between the $f_0$ and $F_0$ models. For instance (2.7a)–(2.8a), (2.7b)–(2.8b)...

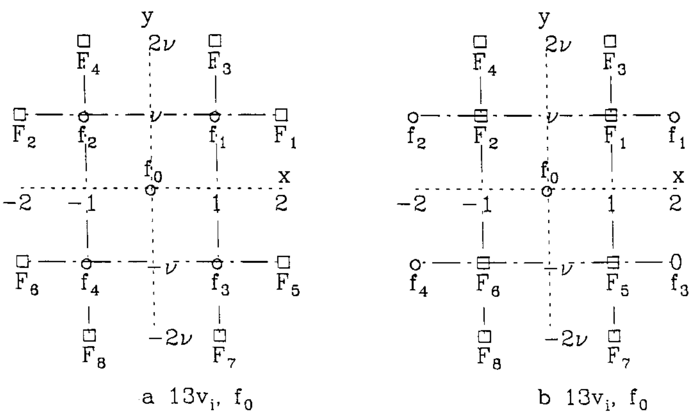



Fig. 3. (a) $13v_i$ model with $f_0$ light rest-particle, $M = 2$ and $v = \sqrt{5}$. (b) $13v_i$ model with $f_0$ light rest-particle, $M = 2$ and $v = \sqrt{8}$.

From Lemma 1 we still have $\sum L_i = \sum l_i = 0$ (no spurious invariant for the set with $f_0$ or $F_0$). For the $13v_i$ models (Figs. 3), as for the case of the $11v_i$ models (Figs. 2a-b-d), the momenta are in two intervals parallel to the $x$-axis (the $y$-axis) with opposite values, $y = \pm v$ $(x = \pm 1)$ and, as we shall prove, without spurious invariant for the set without $f_0$ or $F_0$. We call $X_{\pm v}$ and $Y_{\pm 1}$ the sums of the left-hand side of the evolution equations referring to these intervals. We call $Z$ the sum of the evolution equations for the densities in the external layer of the model, for instance $Z = \sum L_i = 0$ in Fig. 3a and $Z = L_{3,4,7,8} + l_{1,2,3,4}$ in Fig. 3b. However, concerning the general results without explicit collisions (Lemma 2), here they are in four intervals (three for $11v_i$) and without the rest particle. So we consider only the energy relation.

**Lemma 4.** For the evolution equations of the external layer we get $Z = 0$ and either $Y_{1,-1} = 0$ in (2.qa) or $X_{v,-v} = 0$ in (2.qb) models.

We give the proof for two (2.qa) models. In Fig. 3a we get $(4 + v^2)\, Z/M + (1 + v^2)\, Y_{1,-1} = 0$, $Z = 0$ and in Fig. 3b $(1 + v^2) \sum L_i/M + (4 + v^2)\, Z = 0$, $\sum L_i = 0 = Y_{1,-1}$. In (2.8a) we have $Y_{1,-1} = \sum l_i = 0$ and $Z = \sum l_i = 0$ in (2.9a). For the (2.qb) models the change is that due to the "rotation by $\pi/2$," then $Y_{1,-1} = 0$ becomes $X_{v,-v} = 0$.

With Lemmas 1–4 we have three linear relations and 5 physical invariants (adding the two momentum relations). For the mass of the species without $f_0$ or $F_0$, like in the $9v$, $11v$ models, we have the geometrical structure of momenta parallel to the two axes with opposite values. *For the (2q.qa, b) models we have 15 collisions, however only 7 are sufficient for the proof.*

**Lemma 5.** No spurious invariants for the $(L_i)$ sets of Figs. 3a–b with $f_0$.

It is sufficient that all $F_i$'s be linked in the loss and gain terms of particular collisions with the only possible linear combination $\sum L_i = 0$. For the 4 collisions with exchange of energy we have 4 couples with $L_{i,j} = 0$. There exist 2 pairs of $F_k$, $F_l$ having collisions with sums of momenta along the $x$ axis. For these collisions we deduce two triplets with $L_{i,j,k} = 0$. Finally there exist collisions $\Gamma$ with sum of momenta along y where in the loss and gain terms we have only particles of the first and second triplets.

**As Illustration** *we begin with the (2.6a), Fig. 3a model*:

$$\Gamma_1 = c(f_0 F_3 - f_2 F_1), \qquad \Gamma_2 = \bar{c}(f_3 F_1 - f_1 F_5), \qquad \Gamma_3 = c(f_0 F_7 - f_4 F_5)$$

$$\Gamma_4 = c(f_0 F_4 - f_1 F_2), \qquad \Gamma_5 = \bar{c}(f_3 F_2 - f_1 F_6), \qquad \Gamma_6 = c(f_0 F_8 - f_3 F_6)$$

With $\Gamma = \tilde{c}(f_2 F_3 - f_1 F_4)$ we get $L_{1,3,5,7} = -\Gamma$, $L_{2,4,6,8} = \Gamma$ and $\sum L_i = 0$. As an application we consider the model (2.9a) with $F_0$. Although the numerical values for $v$ are different we have the formal symmetry with the Lemma 5 (2.6a) model: $\vec{p}_i \rightleftharpoons \vec{P}_i$ and collisions with $F_i \rightleftharpoons f_i$ and there is no spurious mass relation for the light species $(l_i)$. *We go on with the (2.7a), Fig. 3b model*:

$$\Gamma_1 = c(f_0 F_3 - f_1 F_2), \qquad \Gamma_2 = \bar{c}(f_3 F_2 - f_1 F_6), \qquad \Gamma_3 = c(f_0 F_7 - f_3 F_6)$$

$$\Gamma_4 = c(f_0 F_4 - f_2 F_1), \qquad \Gamma_5 = \bar{c}(f_3 F_1 - f_1 F_5), \qquad \Gamma_4 = c(f_0 F_8 - f_4 F_5)$$

With $\Gamma = \tilde{c}(F_2 F_3 - F_1 F_4)$ we get $L_{2,3,6,7} = -2\Gamma$, $L_{1,4,5,3} = 2\Gamma$ and $\sum L_i = 0$. Comparing the present model (2.7a) with the model (2.8a) for $F_0$, we still have the formal symmetry $\vec{p}_i \rightleftharpoons \vec{P}_i$ and we deduce only $\sum l_i = 0$ for the $(l_i)$.

## 3. SYMMETRIC $qv_i \cup \hat{q}v_i$, $q = 9, 11, 13$, MODELS (FIG. 4)

To the previous $qv_i$ models (tips of the momenta parallel to the $x$ and $y$ axes) with $(x, y)$ momenta we add, with a rotation of $\pi/2$, new models $\hat{q}v_i$ $(q = 9, 11, 13)$ with momenta components $(y, x)$. As we shall see, contrary to the momentum conservations, there is no spurious invariant for the mass conservations of the light and heavy species. We have two cases:

(i) either $v \neq 1 \neq 2$, $v$ not rational and only the particle at rest is common to the two models $qv_i \cap \hat{q}v_i$ giving $qv_i \cup \hat{q}v_i = (2q-1)\, v_i = 17v_i$, $21v_i$, $25v_i$.

(ii) $v = 1$ or $2$ where we have in common $11v_i \cap \widehat{11}v_i$ the particle at rest and four other, $11v_i \cup \widehat{11}v_i = (22-5)\, v_i = 17v_i$. *In this case, contrary to (i) there is no spurious momentum invariant.*

Let us define, for the $qv_i$, $\hat{q}v_i$ models, the three classes of collisions $\mathrm{Col}_{qv_i}$, $\mathrm{Col}_{\hat{q}v_i}$ with only $f_i, F_i \in qv_i$, and $\hat{f}_i, \hat{F}_i \in \hat{q}v_i$, respectively, and $\mathrm{Col}_{qv_i + \hat{q}v_i}$ the collisions mixing the particles of the two models $qv_i$ and $\hat{q}v_i$.

### 3.1. Mass Conservation

Lemma 1 is still valid with $\sum l_i = \sum L_i = 0$ without spurious invariants for the species with a particle at rest.

**Lemma 6.** The mass species without rest particle has no spurious invariant.

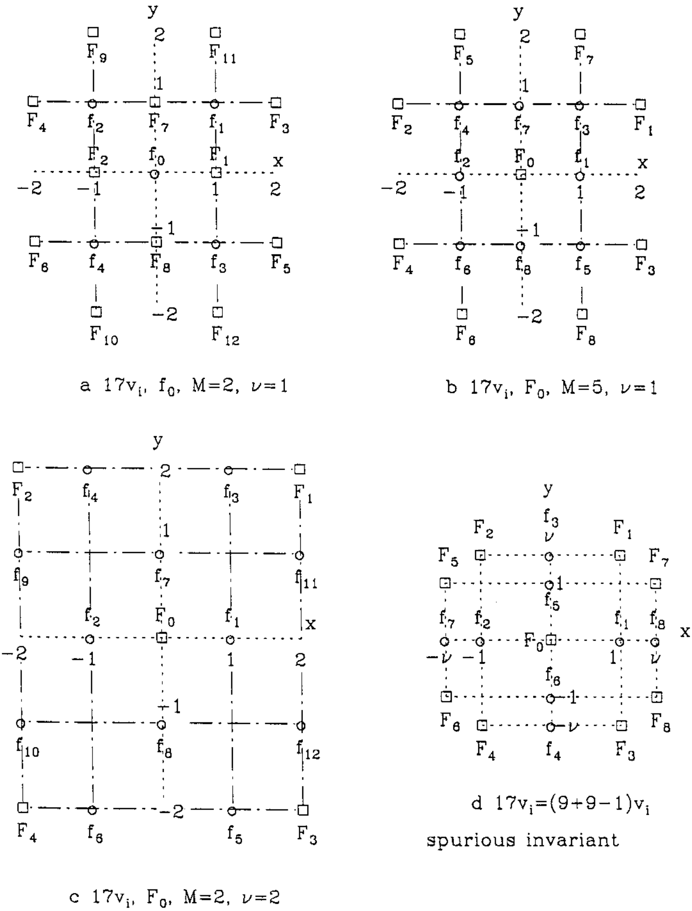
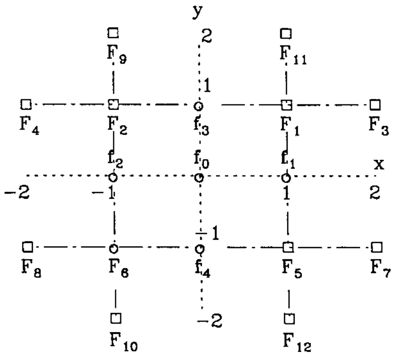



Fig. 4. (a) Symmetric $17v_i = (11 + 11 - 5)\, v_i$ model with $f_0$ light rest-particle, $M = 2$ and $\nu = 1$. (b) Symmetric $17v_i = (11 + 11 - 5)\, v_i$ model with $F_0$ heavy rest-particle, $M = 5$ and $\nu = 1$. (c) Symmetric $17v_i = (11 + 11 - 5)\, v_i$ model with $F_0$ heavy rest-particle, $M = 2$ and $\nu = 2$. (d) Symmetric $17v_i = (9 + 9 - 1)\, v_i$ model with $F_0$ heavy rest-particle, $\nu$ not rational and spurious momentum conservations. (e) Symmetric $17v_i = (11 + 11 - 5)\, v_i$ model with $f_0$ light rest-particle, $M = 3$ and $\nu = 1$.

All the particles of the $qv_i$ model, with momenta $(x, y)$, are associated with other particles with opposite momenta $(-x, -y)$. For the $\hat{q}v_i$ models, similar pairs with $(y, x)$ and $(-y, -x)$ and the same energy. In $\mathrm{Col}_{qv_i + \hat{q}v_i}$ we have such collisions where in the loss and gain terms we have pairs of particles belonging either to the $qv_i$ or to the $\hat{q}v_i$ models (all particles of $qv_i$, $\hat{q}v_i$ belong to such pairs). For the subset of collisions with all the pairs of $qv_i$ particles for the species without a particle at rest, the corresponding




e 17v$_i$, f$_0$, M=3, $\nu$=1

Fig. 4. (*Continued*)

collision terms vanish only if they include the associated pairs of $\hat{q}v_i$ of the same species. Let us start with $F_0$ and consider $\Sigma = \sum l_i$ for all $f_i \in qv_i$. Then $\Sigma = 0$ for either $\mathrm{Col}_{qv_i}$ or $\mathrm{Col}_{\hat{q}v_i}$. For the above $\mathrm{Col}_{qv_i + \hat{q}v_i}$ collisions, $\Sigma = 0$ only if we add $\hat{f}_i \in \hat{q}v_i$ which are all associated to these collisions. Finally for the mass, $\Sigma = 0$ only if it includes all $(f_i, \hat{f}_i) \in qv_i \cup \hat{q}v_i$ and so without spurious invariant.

## 3.2. Momentum Conservation

We recall a well known result. Let $\mathrm{Col} \simeq F_k F_l - F_m F_p$ (or $F_k f_l - F_m f_p$, or $f_k f_l - f_m f_p$). A necessary condition for a collision $(x_k, y_k) + (x_l, y_l) \leftrightarrow (x_m, y_m) + (x_p, y_p)$ to exist is that the contribution for the momentum along the $x$-axis ($y$-axis) vanishes. Consequently for DVMs the momentum conservations are satisfied when all particles are considered. The problem is whether they can be satisfied in two subsets. We will find a virus for such spurious invariants. Due to the symmetry of the model, it is sufficient to study the $x$-axis momentum relation. For instance let us define $J_{qv\ldots}$, $J_{\hat{q}v_i}$, $J_{qv_i \cup \hat{q}v_i}$ the momentum relation along the $x$-axis for respectively the $qv_i$, $\hat{q}v_i$, $qv_i \cup \hat{q}v_i$ models.

*For the symmetric models, we will find* $J_{qv_i} = 0, \neq 0$, $J_{\hat{q}v_i} = 0, \neq 0$ *depending on whether only one or five particles are in common.*

### 3.2.1. Only One Common Particle at Rest: $\neq 1$, $\neq 2$, Not Rational

**Lemma 7.** For $\nu$ not rational, $J_{qv_i \cup \hat{q}v_i} \equiv J_{qv_i} + J_{\hat{q}v_i} \equiv 0$ with the three quantities being zero. A spurious momentum invariant exists.

We have verified these properties for the symmetric $17v_i$, $21v_i$, $25v_i$ models with $q = 9$, 11, 13 (respectively 17, 25, 31 collisions). For a general proof we recall that the collisions for the composed model are the sum of the collisions $\mathrm{Col}_{qv_i}$, $\mathrm{Col}_{\hat{q}v_i}$ and new ones $\mathrm{Col}_{qv_i+\hat{q}v_i}$ connecting the two models. Let us define $(f_i, F_i) \in qv_i$ and $(\hat{f}_i, \hat{F}_i) \in \hat{q}v_i$. It is sufficient to verify that these new collisions vanish in the momentum relations of the two components.

*First in such collisions we cannot have pairs $f_i$, $\hat{F}_j$ or $\hat{f}_i$, $F_j$ in the loss and gain terms.* Let us consider a sum of momenta associated to the two models with $q=9, 11, 13$. For the $x$-coordinate we have $s_1+s_2\nu$ with $s_i=$ either $0$, $\pm 1$, $\pm 2$ and $\nu$ not rational. For a collision with another pair of the same type we have $s_3+s_4\nu$. The equality requires $s_j=s_{j+2}$ and the two pairs are identical.

*Second we consider pairs belonging to either $qv_i$ or $\hat{q}v_i$.* For instance pairs $f_iF_j$ and $\hat{f}_k\hat{F}_l$ or $F_iF_j$ and $\hat{F}_k s\hat{F}_l$ or $f_if_j$ and $\hat{f}_k\hat{f}_l$. Along the $x$-axis we must have the equality $(s_1+s_2, (s_3+s_4)\,\nu) \equiv (\nu(s_5+s_6), s_7+s_8)$, which is possible only if $s_j+s_{j+1}=0$, $j=1, 3, 5, 7$. Such pairs give zero for the sum of momenta which means, for the $qv_i$, $\hat{q}v_i$, only heavy or light particles with opposite momenta. Such collisions give, to the momentum conservations, a zero contribution to both $J_{qv_i}$, $J_{\hat{q}v_i}$, and $J_{qv_i \cup \hat{q}v_i}$. *We have four independent linear relations for the two momentum relations. Adding the two mass relations for the heavy and light species, there is at least one spurious invariant.*

**As illustration** of Lemma 7, we write down the relations for the $17v_i = 9v_i \cup \hat{9}v_i$ model, Fig. 4d. From the 6 collisions $\mathrm{Col}_{9v_i}$ (without $\Omega$) for the $F_i$, $f_i$, written in (2.9), we deduce the 6 collisions for $\mathrm{Col}_{\hat{9}v_i}$ with $F_{i+4} := \hat{F}_i$, $f_{i+4} := \hat{f}_i$. For $\mathrm{Col}_{9v_i+\hat{9}v_i}$, we get 5 collisions with pairs of opposite momenta:

$$\Omega_1 = b(F_1F_4 + F_2F_3 + \hat{F}_1\hat{F}_4 - 3\hat{F}_2\hat{F}_3)$$

$$\Omega_2 = b(F_2F_3 + \hat{F}_1\hat{F}_4 + \hat{F}_2\hat{F}_3 - 3F_1F_4)$$

$$\Omega_3 = b(\hat{F}_1\hat{F}_4 + \hat{F}_2F_3 + F_1F_4 - 3F_2F_3)$$

$$-\sum_1^3 \Omega_i = b(\hat{F}_2\hat{F}_3 + F_1F_4 + F_2F_3 - 3\hat{F}_1\hat{F}_4) \tag{3.1}$$

$$\Lambda_1 = d(f_1f_2 - \hat{f}_1\hat{f}_2), \qquad \Lambda_2 = \bar{d}(f_3f_4 - \hat{f}_3\hat{f}_4)$$

$$J_{9v_i} = L_{1,2} - L_{3,4} + l_1 - l_2 = 0, \qquad J_{\hat{9}v_i} = \nu[\hat{L}_{3,4} - \hat{L}_{1,2} + \hat{l}_4 - \hat{l}_3] = 0$$

$$J_{17v_i} \equiv J_{9v_i} + J_{\hat{9}v_i} = 0$$

$\mathrm{Col}_{9v_i}$ and $\mathrm{Col}_{\hat{9}v_i}$ give zero for both $J_{9v_i}$, $J_{\hat{9}v_i}$. For $\mathrm{Col}_{9v_i+\hat{9}v_i}$ with opposite momenta, $F_iF_j$, $f_if_j$... give $J_{9v_i}=J_{\hat{9}v_i}=0$. We have two momentum conservations along the $x$-axis and two other ($x \rightleftharpoons y$ symmetry of the model) along the $y$-axis. This symmetric $17v_i$ model has one spurious invariant.

### 3.2.2. Five Common Momenta ($v=1, 2$) and $17v_i = 11v_i \cup \widehat{11}v_i$ Models

To our two previous $17v_i$ physical models:[3] $M=2$, $v=1, 2$ Figs. 4a–c we add two new ones: $M=5, 3$, $v=1$ Figs. 4b–e. For these four symmetric models we have the 5 physical conservation equations. For collisions like those in Lemma 7, with pairs of opposite momenta belonging to one of the two models, we still have $J_{11v_i}=J_{\widehat{11}v_i}=0$. We consider collisions with exchange of energy and, we remark, with only one of the common momenta. We deduce $J_{11v_i}\neq 0$, $J_{\widehat{11}v_i}\neq 0$. *The main point is that now the momentum relations are not the sum of the corresponding ones for the two models because an additional relation arises from the four common nonzero momenta.*

**Lemma 8.** For $qv_i \cup \hat{q}v_i$ models (any $q$ value) with $f_0$, $F_0$ and other $f_i$ (or $F_i$) common, we have $J_{qv_i}\neq 0$, $J_{\hat{q}v_i}\neq 0$ and no spurious momentum relation.

For the densities $F_i$, $f_i \in 11v_i$ and $\hat{F}_i$, $\hat{f}_i \in \widehat{11}v_i$, we first consider a collision with energy exchange with two of these common momenta (($\eta_1 x$, $\eta_2 x$), $\eta_i^2=1$ with the same energy). For $F_0 f_i - F_k f_j$ (or $f_0 F_i - f_k F_j$) with ($f_i$, $f_j$) (or ($F_i$, $F_j$)) common we cannot satisfy the energy conservation in loss and gain terms. Second, for such collision, we consider only one of the common densities $f_i \in qv_i \cap \hat{q}v_i$ for $\Gamma \simeq f_0 f_i - F_k f_l$ and another $\hat{f}_i$ in the associated collision term (rotated by $\pi/2$) $\hat{\Gamma} \simeq f_0 \hat{f}_i - \hat{F}_j \hat{f}_l$. In $\Gamma$, with only one of the common $f_i$ we deduce that only ($f_i$, $F_k$, $f_l$, $\hat{f}_i$) $\in qv_i$ and only ($f_i$, $\hat{F}_j$, $\hat{f}_l$, $\hat{f}_i$) $\in \hat{q}v_i$. Consequently the contributions of $\Gamma$ and $\hat{\Gamma}$ to $J_{qv_i}$ are respectively zero and const.$\hat{\Gamma}$. Similarly these contributions to $J_{\hat{q}v_i}$ are respectively const.$\Gamma$ and zero. We have similar results if, for $f_0$, the common momenta correspond to $F_i$ or if starting with $F_0$, the common momenta correspond to either $f_i$ or $F_i$. For $qv_i = 11v_i$, we write the four models and notice that $J_{17v_i}\neq J_{11v_i}+J_{\widehat{11}v_i}$:

$$v=1,\ M=2,\ \text{Fig. 4a (37 collisions)},\ f_i,\ i=0, 1, 2, 3, 4 \in 11v_i \cap \widehat{11}v_i,$$

$$J_{17v_i}=J_{11v_i}+J_{\widehat{11}v_i}-l_{1,3}^{2,4}=0, \qquad l_{i,j}^{m,n}:=l_{i,j}-l_{m,n} \tag{3.2}$$

$v=2$, $M=2$, Fig. 4c, (33 collisions), $F_i$, $i=0, 1, 2, 3, 4$ common,

$$J_{17v_i}=J_{11v_i}+J_{\widehat{11}v_i}-l_{1,3}^{2,4}=0, \qquad L_{i,j}^{m,n}=L_i+L_j-L_m-L_n \tag{3.3}$$

$v=1$, $M=5$, Fig. 4b (37 collisions), $F_0, f_i$, $i=3, 4, 5, 6$ common,

$$J_{17v_i}=J_{11v_i}+J_{\widehat{11}v_i}-l_{3,5}^{4,6}=0 \tag{3.4}$$

$v=1$, $M=3$, Fig. 4e, (33 collisions), $f_0$, $F_i$, $i=1, 2, 5, 6$ common,

$$J_{17v_i}=J_{11v_i}+J_{\widehat{11}v_i}+L_{2,6}^{1,5} \tag{3.5}$$

**Illustrations of Lemma 8** for $q=11$: (i) We begin with Eq. (3.2), $f_0$, $\hat{F}_i=F_{i+6}$, $\Gamma\simeq f_0F_3-f_1F_1$ and $\hat{\Gamma}\simeq f_0\hat{F}_3-\hat{f}_1\hat{F}_1\simeq f_0F_9-f_2F_7$. We write the contribution of $\Gamma$, $\hat{\Gamma}$ to the momentum relations along the $x$-axis:

$$J_{11v_i}=2L_{3,5}^{4,6}+L_1-L_2+l_{1,3}^{2,4}=-\hat{\Gamma}\neq 0, \qquad J_{\widehat{11}v_i}=L_{11,12}^{9,10}+l_{1,3}^{2,4}=\Gamma\neq 0$$

(ii) We go on with the Eq. (3.4) model with $F_0$, $\hat{F}_i=F_{i+4}$, $\Gamma\simeq F_0f_3-F_1f_2$, $\hat{\Gamma}\simeq F_0\hat{f}_3-\hat{F}_1\hat{f}_2\simeq F_0f_4-F_5f_7$ and write the contributions:

$$J_{11v_i}=2L_{1,3}^{2,4}+l_1-l_2+l_{3,5}^{4,6}=\hat{\Gamma}\neq 0, \qquad J_{\widehat{11}v_i}=L_{7,8}^{5,6}+l_{3,5}^{4,6}=-\Gamma\neq 0$$

(iii) We consider the Eq. (3.3) model with $F_0$, $\hat{f}_i=f_{i+6}$, $\Gamma\simeq F_0f_3-F_1f_2$, $\hat{\Gamma}\simeq F_0\hat{f}_3-\hat{F}_1\hat{f}_2\simeq F_0f_9-F_2f_8$ giving the contributions:

$$J_{11v_i}=2L_{1,3}^{2,4}+l_1-l_2+l_{3,5}^{4,6}=2\hat{\Gamma}\neq 0, \qquad J_{\widehat{11}v_i}=2L_{1,3}^{2,4}+l_{11,12}^{9,10}=2\Gamma\neq 0$$

(iv) We finish with the Eq. (3.5) model with $f_0$, $\Gamma\simeq f_0F_3-F_1f_1$, $\hat{\Gamma}\simeq f_0\hat{F}_3-\hat{F}_1\hat{f}_1\simeq f_0F_9-F_2f_3$ giving the contributions:

$$J_{11v_i}=2L_{3,7}^{4,8}+l_1-l_2+L_{1,5}^{2,6}=-\hat{\Gamma}\neq 0, \qquad J_{\widehat{11}v_i}=L_{11,12}^{9,10}+L_{1,5}^{2,6}=\Gamma\neq 0$$

We verify with (3.2–3.5) that the contributions of $\Gamma$, $\hat{\Gamma}$ to $J_{17v_i}$ are zero.

**As illustration of Lemma 6**, *we verify that the mass of the species without particle at rest (for brevity only for the (3.5)–(3.9) model with 33 collisions) does not lead to any spurious invariant*. Only 11 collisions are sufficient:

$$\Gamma_1\simeq f_0F_3-f_1F_1, \qquad \Gamma_2\simeq f_0F_4-f_2F_2, \qquad \Gamma_3\simeq f_0F_7-f_1F_5,$$

$$\Gamma_4\simeq f_0F_8-f_2F_6, \qquad \Gamma_5\simeq f_2F_1-f_1F_2, \qquad \hat{\Gamma}_1\simeq f_0F_9-f_3F_2,$$

$$\hat{\Gamma}_2\simeq f_0F_{10}-f_4F_6, \qquad \hat{\Gamma}_3\simeq f_0F_{11}-f_3F_1$$

$$\hat{\Gamma}_4\simeq f_0F_{12}-f_4F_5, \qquad \hat{\Gamma}_5\simeq f_4F_2-f_3F_6, \qquad \hat{\Gamma}_6\simeq f_4F_1-f_3F_5$$

We eliminate successively $\Gamma_1$, $\Gamma_5$, $\Gamma_2$, $\hat{\Gamma}_1$, $\hat{\Gamma}_3$, $\hat{\Gamma}_6$, $\hat{\Gamma}_5$, $\hat{\Gamma}_2$, $\hat{\Gamma}_3$, $\hat{\Gamma}_4$ in $L_{1,3}$, $L_{1,2,3}$, $L_{1,2,3,4}$, $L_{1,2,3.4,9}$, $L_{1,2,3,4,9,11}$, $L_{1,2,3,4,5,9,11}$, $L_{1,2,3,4,5,6,9,11}$, $L_{1,2,3,4,5,6,9,10,11}$, $\sum L_i - L_{12}$, $\sum L_i - L_{8,12}$, and finally $\Gamma_4$ in $\sum_1^{12} L_i = 0$.

## 4. ONE-DIMENSIONAL $7v_i$ AND SYMMETRIC $13v_i$ PLANAR MODELS

### 4.1. $qv_i = 7v_i$ Model (Fig. 5) with Momenta Along the $x$-Axis

We start with the binary collision $F_0 f_3 - f_2 F_1$ valid for any value $M > 1$:

$$\vec{P}_0 + \vec{p}_3(M+1, 0) = \vec{p}_2(1-M, 0) + \vec{P}_1(2M, 0) \to M \geqslant 2 \tag{4.1}$$

add $\vec{p}_i + \vec{p}_{i+1} = \vec{P}_j + \vec{P}_{j+1} = 0$, $i = 1, 3$, $j = 1$ with a new collision $F_0 f_4 - F_2 f_1$. With 7 densities $F_i$, $i = 0, 1, 2$, $f_i$, $i = 1, 2, 3, 4$ and only two collisions this model (only 4 physical invariants for a one-dimensional model), has necessarily one spurious invariant. We write the collisions and the spurious invariant for the mass of the species without a particle at rest,

$$\Gamma_1 = c(F_0 f_3 - F_1 f_2), \qquad \Gamma_2 = c(F_0 f_4 - F_2 f_1), \qquad \to l_{2,3} = l_{1,4} = 0 \tag{4.2}$$

while the three other invariants are physical: mass for the heavy particles, momentum along the $x$-axis and energy. The previous test for planar models (tips of the momenta not parallel to the $y$ axis) cannot be applied for a model along $x$. However we also have missing collisions for the two subsets of the light species. *A medicine*, with a modification of the philosophy underlying the continuous Boltzmann equation,[5] is the inclusion of multiple collisions:

$$\Lambda = \bar{a}(f_1^{M+1} f_4^{M-1} - f_2^{M+1} f_3^{M-1}), \qquad 2M\Lambda = l_{2,3} = -l_{1,4} \to \text{only} \sum l_i = 0 \tag{4.3}$$

### 4.2. $7v_i \cup \hat{7}v_i = 13v_i$ Symmetric Planar Model (Fig. 5) with $M \geqslant 2$

This gives, for $M = 2$, 5, the two $13v_i$ models of ref. 1. We add the $\hat{q}v_i = \hat{7}v_i$ model with momenta along the $y$-axis, letting $\hat{f}_j = f_{i+4}$, $\hat{F}_j = F_{i+2}$.

*We begin with binary collisions* and add 5 collisions, two with $\hat{f}_i$, $\hat{F}_i$ in $\hat{\Gamma}_i = \Gamma_{i+2}$ and three other from $\Omega = b(F_1 F_2 - F_3 F_4)$, $\Lambda_i = a(f_i f_{i+1} - f_{i+4} f_{i+5})$, $i = 1, 3$. With 13 densities and 7 collisions we have at least one

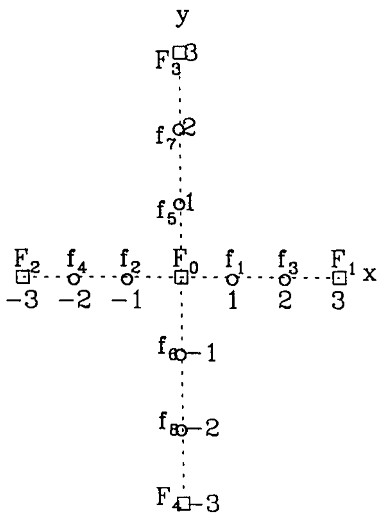



Fig. 5. $M=3$, $7v_i$ and Symmetric $13v_i=(7+7-1)\,v_i$ model with $F_0$ heavy rest-particle and without spurious heavy mass conservations with ternary collisions.

spurious invariant. We find two spurious mass relations for the light species:

$$\Lambda_{1,2}=-l_{1,4}=-l_{2,3}=l_{5,8}=l_{6,7}, \qquad l_{1,4}+l_{5,8}=l_{1,4}+l_{6,7}=l_{1,4}+l_{2,3}=0 \tag{4.4}$$

*We go on with multiple collisions*, adding $\hat{\Lambda}=\bar{a}(f_5^{M+1}f_8^{M-1}-f_6^{M+1}f_7^{M-1})$:

$$\Lambda_{1,2}=-l_{1,4}-2M\Lambda=-l_{2,3}+2M\Lambda=l_{5,8}+2M\hat{\Lambda}=l_{6,7}-2M\hat{\Lambda} \tag{4.5}$$

Only the physical mass relation $\sum_1^8 l_i=0$, for the light species, remains.

## 5. $15v_i$ AND SYMMETRIC $15v_i \cup \widehat{15}v_i=25v_i$ MODELS (FIGS. 6 and 7)

The $25v_i$ ($M=2, 5$) model is important. First, it was one of the two explicit models presented in ref. 1. Second, some people, with powerful computers found that it has spurious invariants and this negative result was confirmed in ref. 5. Here we show that only for $M=5$ spurious invariant exists (eliminated with multiple collisions) while the $M=2$ model, with binary collisions, has no spurious invariant. In fact the virus for $M=5$ can be detected at the $qv_i=15v_i$ level. Like previously for some

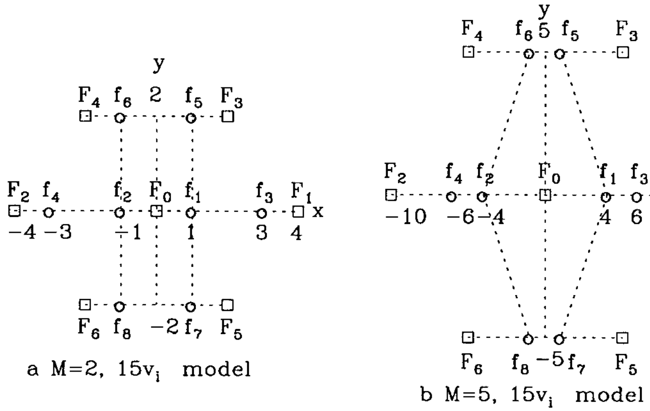



Fig. 6. (a) $15v_i$ model with $F_0$ heavy rest-particle $M=2$ and without spurious invariant. (b) $15v_i$ model with $F_0$ heavy rest-particle $M=5$ and with spurious invariant.

planar $qv_i$ models, we have (contrary to $M=2$) tips of the momenta not parallel to the $y$-axis. These different features between $M=2$ and $M=5$ are easily seen in the Figs. 6–7. We study the $15v_i$ and briefly the $25v_i$ models while a complete proof is given in ref. 7.

## 5.1. Planar $15v_i$, $M \geqslant 2$ Models (Fig. 6; $M=2, 5$)

First we start with two collisions with exchange of energy for all $M \geqslant 2$:

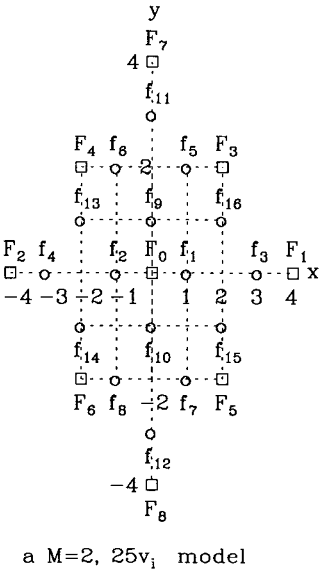
$$\begin{aligned} \Gamma_1 &= F_0 f_3 - f_2 F_1, \qquad \vec{p}_3(M+1, 0) = \vec{p}_2(1-M, 0) + \vec{P}_1(2M, 0) \\ \Omega_1 &= F_0 F_1 - F_3 F_5, \qquad \vec{P}_1(2M, 0) = \vec{P}_3(M, M) + \vec{P}_5(M, -M) \end{aligned} \tag{5.1}$$

Second, from (5.1), we deduce another collision with exchange of energy:

$$\Gamma_3 = F_0 f_5 - F_3 f_2, \qquad \vec{p}_5(1, M) = \vec{P}_3(M, M) + \vec{p}_2(1-M, 0) \tag{5.2}$$

With $(\eta_1 x, \eta_2 y)$, $\eta_i^2 = 1$ we obtain the $qv_i = 15v_i$ model: $\vec{p}_6(-1, M)$, $0 = \vec{p}_i + \vec{p}_{i+1} = \vec{P}_1 + \vec{P}_{i+1}$, $i = 1, 3, 6$, $j = 1, 4$, $\vec{P}_6 + \vec{P}_3 = \vec{p}_8 + \vec{p}_5 = 0$.

Lemma 1 is still valid with only $\sum_0^6 L_i = 0$. *Geometrically we notice* that momenta along two intervals parallel to the $y$-axis with $x = \pm 1$ exist only for $M=2$. From the previous sections we expect for $M \neq 2$ (binary collisions) a spurious mass invariant (medicine with multiple collisions) for the light species.



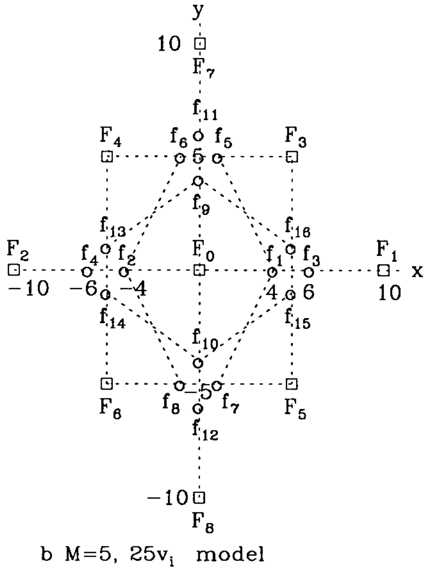




Fig. 7. (a) Symmetric $25v_i = (15 + 15 - 5)\, v_i$ model with $F_0$ heavy rest-particle, $M = 2$ and without spurious invariant. (b) Symmetric $25v_i = (15 + 15 - 5)\, v_i$ model with $F_0$ heavy rest-particle, $M = 5$ and with spurious invariant.

**Lemma 9.** For the light mass species, there is no spurious invariant for $M = 2$ with binary collisions and $M \neq 2$, integer, with multiple collisions.

For the collisions with exchange of energy we remark that $f_i$, $i = 1, 2$ and respectively $f_{5-i}$, $f_{7-i}$, $f_{9-i}$ are in the loss and gain terms and we deduce two quadruplets: $l_{i,\,5-i,\,7-i,\,9-i} = 0$. In these two subsets we remark that the momenta of $f_{7-i}$, $f_{9-i}$ (only ones not along the $x$-axis) have the same $x$ but opposite $y$ values. Consequently for the collisions with either sums of momenta along the $x$-axis or opposite momenta, they are in loss and gain terms and we still have $l_{i,\,5-i,\,7-i,\,9-i} = 0$. For the collisions with sums of momenta along the $y$-axis (with the two quadruplets in loss and gain terms), they are possible for $M = 2$ (binary collisions and tips of momenta parallel to the $y$-axis) or for $M$ integer $\neq 2$ (multiple colisions) eliminating the spurious invariant.

**As Illustration**, we write 6 (among 16) binary collisions common to $M = 2$, $\neq 2$ and add two other for $M = 2$ and for $M \neq 2$, integer (multiple).

$$\Gamma_1 = F_0 f_4 + F_4 f_8 + F_6 f_6 - 3F_2 f_1, \qquad \tilde{\Gamma}_2 = F_4 f_8 + F_6 f_6 + F_2 f_1 - 3F_0 f_4$$

$$\tilde{\Gamma}_4 = F_6 f_6 + F_2 f_1 + F_0 f_4 - 3F_4 f_8$$

$$\Gamma_2 = F_0 f_3 + F_3 f_7 + F_5 f_5 - 3F_1 f_2, \qquad \tilde{\Gamma}_3 = F_5 f_5 + F_1 f_2 + F_0 f_3 - 3F_3 f_7$$
$$\tilde{\Gamma}_5 = F_3 f_7 + F_5 f_5 + F_1 f_2 - 3F_0 f_3,$$
$$\Lambda_1 = f_5^{M-1} f_2 - f_6^{M-1} f_1, \qquad \Lambda_2 = f_7^{M-1} f_2 - f_8^{M-1} f_1 \tag{5.3}$$

$\Gamma_i$, $\tilde{\Gamma}_{i+1}$, $\tilde{\Gamma}_{i+3}$ disappear in $l_{i,5-i,7-i,9-i}$, $i=1,2$ leading to: $M\Lambda_{1,2} = l_{1,4,6,8} = -l_{2,3,5,7} \rightarrow$ only light mass $\sum_1^8 l_i = 0.$ for $\Lambda_1 \neq 0$, $\Lambda_2 \neq 0$. For $M=2$ ($M \neq 2$), binary (multiple) collisions we only have the light mass relation. If we restrict to binary collisions for $M \neq 2$, then $\Lambda_1 = \Lambda_2 = 0$ and we have one spurious mass invariant $l_{2,3,5,7} = l_{1,4,6,8} = 0$.

## 5.2. Symmetric $15v_i \cup \widehat{15}v_i = 25v_i$, $M \geqslant 2$, Models (Fig. 7; $M = 2, 5$)

Adding the $\widehat{15}v_i$ model, we obtain the $25v_i$ model (5 common particles) for $M \geqslant 2$. We have found 46 (42) binary collisions for $M=2$ ($M \neq 2$) while we could also add 4 multiple collisions for $M \neq 2$. Lemmas 1–8 are still valid for the mass of the heavy particles and the momentum relations (without spurious invariants). Due to the symmetry of the model, it is sufficient to prove that only 4 physical invariants exist for the $x$-dependent model.

**Lemma 10.** $10+1$ binary collisions are sufficient to prove that there is no spurious invariant for the $25v_i$, $M=2$ mass of the light species $\sum l_i = 0$.

$$i = 1, 2, \qquad \Lambda_{1+i} = f_{4+i}^2 - f_i f_{i+2}, \qquad \tilde{\Lambda}_{2+i} = f_i f_{4+i} - f_9 f_{10+3i}$$
$$\Gamma_i = F_0 f_{3+i} + 2F_{2+i} f_{5+i} - 3F_i f_{3-i}$$
$$\tilde{\Gamma}_i = 2F_{2+i} f_{5+i} + F_i f_{3-i} - 3F_0 f_{3+i} \tag{5.4}$$
$$\tilde{\Lambda}_1 = f_3 f_4 - f_{11}^2, \qquad \tilde{\Lambda}_2 = f_1 f_2 - f_9^2, \qquad \Lambda_1 = f_2 f_5^{M-1} - f_1 f_6^{M-1}$$

First, $l_{11} = \tilde{\Lambda}_1$, $l_{13} = \tilde{\Lambda}_4$, $l_{16} = \tilde{\Lambda}_3$, $l_9 - l_{13,16} = \tilde{\Lambda}_2$, $l_{1,9,16} - l_{13} = \Gamma_2 + \Lambda_{3,1,1}$, $l_{1,9,13,16} + 2l_6 = -\Lambda_3 - \tilde{\Gamma}_2 + 2M\Lambda_1$. Second, $l_{2,9,13} - l_{16} = \Gamma_1 + \Lambda_2 - 2\Lambda_1$, $-l_{2,9,13,16} - 2l_5 = \Gamma_1 + \Lambda_2 + 2M\Lambda_1$. Finally, $2M\Lambda_1 = l_{1,4,5,13,16} + 2l_6 = -[l_{2,3,9,4,13,16} + 2l_5]$ giving only the physical invariant: $\sum l_i = l_{1,2,3,4} + 2l_{5,6,9,13,16} = 0$ if $\Lambda_1 \neq 0$. For $M \neq 2$ and only binary collisions, then $\Lambda_1 = 0$ and we have a spurious mass invariant: $0 = l_{1,4,5,13,16} + 2l_6 = l_{2,3,9,11,13,16} + 2l_5$.

# 6. CONCLUDING REMARKS

In this paper we have discussed the construction of a class of planar DVM for binary mixtures that, *for binary collisions*, we summarize.

(i): As discussed in refs. 1–3 and suggested in the book by Monaco and Preziosi[6] we start with collisions with exchange of energy between the two species $F_0 f_i - F_k f_j$ or $f_0 F_i - f_k f_l$. With the exchange of momenta $(\eta_1 x, \eta_2 y)$, $\eta_i^2 = 1$ we obtain semisymmetric $qv_i$ models (not symmetric in the the exchange between the two axes). For the species with $F_0$ (or $f_0$) all particles are linked to $F_0$ (or $f_0$) and we only have one physical mass conservation. However problems can arise for the mass conservation of the other species or for other physical conservations, like momentum conservations. We notice that for the species without $F_0$ (or $f_0$) they are in both loss and gain terms of these collisions. We retain only two new types of collisions (ii) (or (iii)). We have collisions with sums of momenta along the $x$-axis (or $y$-axis) coming from two momenta with opposite $y$ (or $x$) values.

Finally adding the $(y, x)$ momenta we find the symmetric $qv_i \cup \hat{q}v_i$ models.

(1) We begin with the $qv_i$ $(q = 7, 9, 11, 13, 15)$ models and discuss the possibility of a spurious mass invariant for the species without $F_0$ (or $f_0$):

$7v_i$ with $F_0$, 4 $f_i$ and only 2 collisions of the (i) type (without common $f_i$), we necessarily have two doublets $l_{i,j} = 0$ and one spurious invariant.

$9v_i$ with $F_0$, 4 $f_i$, 4 (i) collisions (common $f_i$) without spurious invariants.

$11v_i$ with $F_0$ (or $f_0$) and 6 $f_i$ (or $F_i$): With 4 collisions of the (i) type and only two $f_i$ (or $F_i$) common we have two triplets $l_{i,j,k} = 0$ (or $L_{i,j,k} = 0$). Adding the (ii) collisions does not change this because the two $f_i$ (or $F_i$) are in the loss and gain terms. If the $qv_i$ momenta are parallel to the $y$ axis with opposite $x$ values (or not) we can (cannot) add collisions of the (iii) type and the two triplets give (do not give) the quadruplet $l_{i,j,k,l} = 0$ without (with) spurious invariant.

$13v_i$ with 8 $f_i$ (or $F_i$) and 4 collisions of the (i) type, without common particle, we get 4 doublets $l_{i,j} = 0$ (or $L_{i,j}$). For the (2.qa, b) models, $q = 6, 7, 8, 9$ the tips of the momenta are either parallel to the $x$ or $y$ axes with opposite values and the (ii), (iii) lead first to two triplets $l_{i,j,k} = 0$ (or $L_{i,j,k} = 0$) and finally to the physical invariant $\sum l_i = 0$ (or $\sum L_i = 0$).

$15v_i$ with $F_0$, 8 $f_i$ and 8 collisions of the (i) type with common $f_i$, leading to two quadruplets $l_{i,j,k,l} = 0$. With $M = 2$ $(M \neq 2)$ we have (do not have) momenta parallel to the $y$-axis, we can (cannot) add collisions of the (iii) type leading to models without (with) spurious invariant.

(2) We go on with the symmetric $qv_i \cup \hat{q}v_i$ models. If $qv_i \cap \hat{q}v_i = 1$ $(\neq 1)$, the momentum relation along the $x$-axis is (is not) the sum of the associated momenta for the $qv_i$, $\hat{q}v_i$ and there is (is not) a spurious momentum invariant. This eliminates the $(9 + 9 - 1 = 17)\, v_i$ and $(13 + 13 - 1 = 25)\, v_i$

models. There remain 5 physical models: 4 $(11+11-5=17)\ v_i$ models (two of them presented in ref. 3) and only one $(15+15-5=25)\ v_i$ model, with $M=2$, presented in ref. 1, mentioned in refs. 2–4 and studied here and in ref. 7). For the other $25v_i$, $M\neq 2$ models, there is not this drawback of spurious momentum invariant but the virus of spurious mass invariant of the starting $15v_i$ model is transmitted automatically.

Concerning the fact that our minimal semisymmetric model is $9v_i$, while in ref. 4 it was our previous $11v_i$ model (ref. 3), the explanation is in our construction of models from a starting energy exchange collision. For particles in the plane we find 4 associated momenta while there are only two for particles along one axis. Such collision cannot exist if the three particles are along the two axes, so the minimal model is with all particles along the same axis leading to our $7v_i$ model (spurious mass relation). The next step is with one particle in the plane and the two other ones along the axes (our $9v_i$ model).

The fact that, with binary collisions we find no spurious relations for the models with $25v_i$, $M=2$, which contradicts both the results of ref. 4 and those obtained with powerful computers, can be explained. If we consider only the common $M=2$, 5 binary collisions (as was done in ref. 1), ignoring the 4 crucial $M=2$ binary collisions, then we find that both models have spurious relations. In our distinction between the two models, the help was given by the two $15v_i$, $M=2$, 5 $(25v_i=15v_i\cup\widehat{15v_i})$. We find for the $15v_i$ models, with $M=5$, the same drawback as for some $11v_i$ models with the tips of the momenta not parallel to the $y$-axis, leading to a spurious relation.

*We must be very careful for similar models with different M values.*

Moreover we mention that our pedestrian method (eliminating successively different collision terms), discarding the particle at rest, can be extended to the three momentum and energy conservations. Any vanishing linear combination, including light and heavy evolution equations, must be a linear combination of these three invariants and the mass of the species without particle at rest. We seek also the minimal number of collisions. Such a study for the $25v_i$, $M=2$ and 5 models was done in ref. 7 with only 15 and 16 collisions.

(3) A crucial property of our semisymmetric $qv_i$ and symmetric $qv_i\cup\hat{q}v_i$ models is that all particles are linked to the particle at rest with energy exchange collisions. Notice that, in order to avoid spurious invariant, our criterion of tips of the momenta along parallels to the two axes is sufficient only for the $qv_i$ models while for the symmetric ones, another criterion (more than one common momenta) was necessary. In ref. 3 we mention the existence of two physical $13v_i$ semisymmetric models,

intermediate between the $11v_i$ and $17v_i$ models. More generally let us define a third class, semisymmetric models intermediate between the $qv_i$ and the symmetric $qv_i \cup \hat{q}v_i$ models where we rotate only some of the $qv_i$ momenta. A great difference with the two previous classes is that all particles are not linked to the particle at rest. Let us consider the $11v_i$, $17v_i$ and $15v_i$, $25v_i$ models (tips of the momenta parallel to the two axes and 4 momenta along the bisectors of the two axes). For the intermediate models we have not found spurious mass invariant. What happens for the possible spurious momentum invariant? There exist intermediate models with this spurious invariant and *some other, that we mention here, without*. (i) *We begin starting with* $11v_i$ where we add two momenta $(0, \pm 1)$ (which are the rotated by $\pi/2$ momenta $(\pm 1, 0)$) and get three physical $13v_i$ models (2 in ref. 3): First in the $17v_i$, $M=5$, Fig. 4b model, we retain only $f_7, f_8$ (eliminate $F_i$, $i=5,..., 8$). Second for the $M=2$, Fig. 4a model, we retain $F_7, F_8$ (eliminate $F_i$, $i=9,..., 12$). Third for the $M=3$, Fig. 4e model, we retain $f_3, f_4$ (eliminate $F_i$, $i=9,..., 12$). (11) *We go on starting with* $15v_i$, $M=2$, *Fig. 6a* for intermediate models between $15v_i$, and $25v_i$, Fig. 7a. First we add 2 momenta $(0, \pm 4)$, (see $F_7, F_8$ in Fig. 7a) and get a $(15+2=17)\, v_i$ model. Second we add 6 momenta $(1, \pm 1)$, $(0, \pm 1)$, $(-1, \pm 1)$ (see $f_{15}, f_{16}, f_{13}, f_{14}$ in Fig. 7a) and get a $(15+6=21)\, v_i$ model. Third and fourth to this $21v_i$ model, we add 2 momenta, either $(0, \pm 4)$ (see $F_7, F_8$ in Fig. 7a) or $(0, \pm 3)$ (see $f_{11}, f_{12}$ in Fig. 7a) and get 2 $(15+8=23)\, v_i$ models.

*The explicit results and proofs will be presented elsewhere.*

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